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LETTER TO THE EDITOR

Spherically symmetric monopoles are smooth

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Abstract. The spherically symmetric monopole solutions, whose existence was rigorously demonstrated by Tyupkin, Fateev and Shvarts are shown to be regular everywhere.

Tyupkin *et al* (1976) showed that the static equations of motion for a vector Yang–Mills field A_μ minimally coupled to the triplet ϕ_a of Higgs scalars in an invariant potential V have solutions of the form

$$A_\mu(x) = \alpha(r)\epsilon_{\mu\rho\sigma}x_\rho L_\sigma, \qquad \phi_a = \beta(r)\hat{x}_a \tag{1}$$

when $V(\phi) = f(|\phi|^2 - \eta^2)^2$ for which the integrated Lagrangian

$$\mathscr{L} = -\int \tfrac{1}{4}\operatorname{Tr}\left(\sum_{\mu,\nu} F_{\mu\nu}^2\right) + \tfrac{1}{2}\sum_{\mu,a} (\nabla_\mu\phi_a)^2 + V(\phi)\,\mathrm{d}^3x$$

is finite where L_σ, $\sigma = 1, 2, 3$ are generators of SU(2) with

$$[L_\rho, L_\sigma] = \epsilon_{\rho\sigma\tau}L_\tau, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

and

$$\nabla_\mu\phi_a = \partial_\mu\phi_a + t(A_\mu)_a^b\phi_b.$$

This gives rigorous justification to heuristic arguments of Polyakov (1974) and t'Hooft (1974). However, it is only shown that $\beta(r) = O(r^{-1/2})$ at the origin, so that these solutions might be singular there.

We have observed (Rawnsley 1977) that this proof can be extended to arbitrary isospin l, where the *ansatz* for ϕ becomes

$$\phi_m(x) = \beta(r)Y_{lm}(\theta, \varphi), \qquad -l \leqslant m \leqslant l \tag{2}$$

and to more general potentials. We shall suppose, for the purpose of proving regularity that $V'(\beta) = U(\beta)\beta$ with U even and $|U(\beta)| \leqslant C\beta^2$ for a constant C, which is certainly true for the quartic potential above.

This proof can be further extended to include a superposition of terms of the form (2) for different isospins provided only that the reduced Lagrangian $\mathscr{L}_A$ has the property that $\delta\mathscr{L}_A = 0$ implies $\delta\mathscr{L} = 0$.

For the *ansatz* (2), $\mathscr{L}_A$ is given by (cf Michel *et al* 1977)

$$\mathscr{L}_A = -4\pi\int_0^\infty (\sigma')^2 + \tfrac{1}{2}r^2(\beta')^2 + \tfrac{1}{2}r^{-2}\sigma^2(\sigma-2)^2 + \tfrac{1}{2}l(l+1)\beta^2(\sigma-1)^2 + r^2V(\beta)\,\mathrm{d}r \tag{3}$$

where $\sigma = r^2\alpha$. The proof of Tyupkin *et al* (1976) uses the derivative terms in this Lagrangian as a Sobolev norm and shows $\mathscr{L}_A$ achieves its maximum for some σ, β in this Sobolev Hilbert space. For $r \neq 0$, $\delta\mathscr{L}_A = 0$ are elliptic second-order equations in one independent variable. Standard results for elliptic operators (Palais 1965) then show σ and β are C^∞ (and even real analytic if V is) except possibly at $r = 0$. From the convergence of $\mathscr{L}_A$ one obtains that σ is finite at $r = 0$ and $\beta = O(r^{-1/2})$. We shall show that for $l = 1$ singularities do not in fact occur and A_μ and ϕ_a given by (1) are C^∞ everywhere. The case of general l is notationally more complicated and will be treated in greater detail elsewhere, but nevertheless the same result holds. In fact we can show $\phi_m = O(r^l)$ at $r = 0$ for the general case (we thank Professor L O'Raifeartaigh for suggesting this should be possible).

The equations $\delta\mathscr{L}_A = 0$ for $l = 1$ are

$$r^2\sigma''(r) = \sigma(r)(\sigma(r)-1)(\sigma(r)-2) + r^2\beta(r)^2(\sigma(r)-1),$$
$$(r^2\beta'(r))' = 2\beta(r)(\sigma(r)-1)^2 + r^2V'(\beta(r)).$$

Introducing $f(r)$, $g(r)$ for $r \neq 0$ with $\sigma(r) = r^2g(r)$, $\beta(r) = rf(r)$ then

$$(r^4f'(r))' = r^4F(f(r), g(r), r) \qquad (r^4g'(r))' = r^4G(f(r), g(r), r);$$
$$F(f, g, r) = 2fg(r^2g-2) + U(rf)f; \qquad G(f, g, r) = g^2(r^2g-3) + f^2(r^2g-1). \tag{4}$$

If we put $\boldsymbol{f} = (f, g)$, $\boldsymbol{F} = (F, G)$ these equations have the form

$$(r^4\boldsymbol{f}'(r))' = r^4\boldsymbol{F}(\boldsymbol{f}(r), r), \qquad r \neq 0. \tag{5}$$

From (4) we see F and G are even functions of their third argument and so we may extend f and g as even functions for $r < 0$ and they still satisfy (5).

We can write

$$F(f(r), g(r), r) = r^{-1}f(r)A(r)$$

with

$$A(r) = 2r^{-1}\sigma(r)(\sigma(r)-2) + rU(\beta(r))$$

which is in $L^2[0, \delta]$, $\delta > 0$ fixed, for $\beta = O(r^{-1/2})$ and $|U(\beta)| \leq C\beta^2$. Also r^3f is in $L^2[0, \delta]$ so $F(f(r), g(r), r)$ is in $L^1[0, \delta]$ and hence the first equation in (4) can be integrated once to give

$$x^4f'(x) - y^4f'(y)$$
$$= \int_y^x r^4F(f(r), g(r), r)\,\mathrm{d}r, \qquad \delta \geq x \geq y > 0$$
$$= \int_y^x r^3f(r)A(r)\,\mathrm{d}r.$$

One may easily show $y^4f'(y) \to 0$ as $y \to 0$ from the convergence of $\mathscr{L}_A$, so letting $y \to 0$,

$$x^4f'(x) = \int_0^x r^3f(r)A(r)\,\mathrm{d}r.$$

The Cauchy–Schwarz inequality then shows that if $|f(r)| \leq C_1r^{-k}$, $k > \frac{1}{2}$, then $|f(r)| \leq C_2r^{-k+\frac{1}{2}}$. By induction, since we have $f(r) = O(r^{-3/2})$ initially, we obtain $f(r) = O(r^{-1/2})$. Similarly $g(r) = O(r^{-1/2})$. Another iteration gives a logarithmic estimate. If, however, the estimates on f and g already obtained are substituted back one can show $r^{-1/2}A(r)$ is in L^2 and get enough room to make one more iteration showing f and g are bounded functions of r near $r=0$. Moreover we have the once-integrated form

$$x^4 f'(x) = \int_0^x r^4 \boldsymbol{F}(f(r), r)\, \mathrm{d}r, \qquad x \neq 0.$$

Making the change of variable $r = xu$, we have

$$f'(x) = x \int_0^1 u^4 \boldsymbol{F}(f(xu), xu)\, \mathrm{d}u, \qquad x \neq 0. \tag{6}$$

Integrating once more:

$$f(x) - f(y) = \int_y^x v \int_0^1 u^4 \boldsymbol{F}(f(uv), uv)\, \mathrm{d}u\, \mathrm{d}v \tag{7}$$

for $\delta \geqslant x \geqslant y > 0$. Since f is bounded, $\tilde{\boldsymbol{F}}(u) = u^{1/2}\boldsymbol{F}(f(u), u)$ is continuous and hence the integral in the right-hand side of (7) converges as $y \to 0$ so $\lim_{y\to 0} f(y)$ exists and we define this to be $f(0)$. Thus f is continuous at 0. Also

$$f(x) - f(0) = x^{3/2} \int_0^1 v^{1/2} \int_0^1 u^{7/2} \tilde{\boldsymbol{F}}(uvx)\, \mathrm{d}u\, \mathrm{d}v$$

so that $f'(0)$ exists and is zero. But $f'(x)$ by (6) tends to zero as $x \to 0$ so $f'(x)$ is continuous, that is $f(x)$ is C^1. Proceeding in this way one may prove by induction that $f(x)$ is C^∞ at $x = 0$ and hence C^∞ everywhere. Finally, since f and g are even there are functions, $\tilde{f}$, $\tilde{g}$ which are C^∞ with $f(r) = \tilde{f}(r^2)$, $g(r) = \tilde{g}(r^2)$ showing that

$$A_\mu(x) = \tilde{g}(r^2)\epsilon_{\mu\rho\sigma} x_\rho L_\sigma, \qquad \phi_a(x) = \tilde{f}(r^2) x_a$$

are C^∞ everywhere, and $\phi_a = O(r)$ at $r = 0$.

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